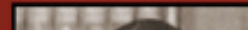
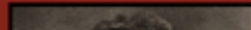


MODERN PHYSICS

FOR SCIENTISTS AND ENGINEERS

SECOND EDITION



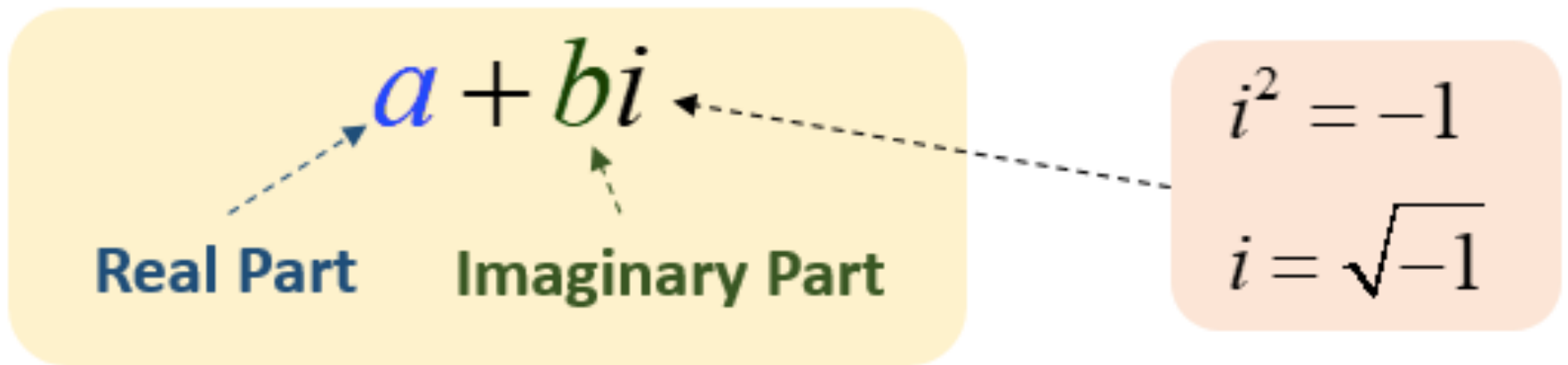
Chapter 2.5 Useful math relations Professor Matthews / UNM



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Complex numbers

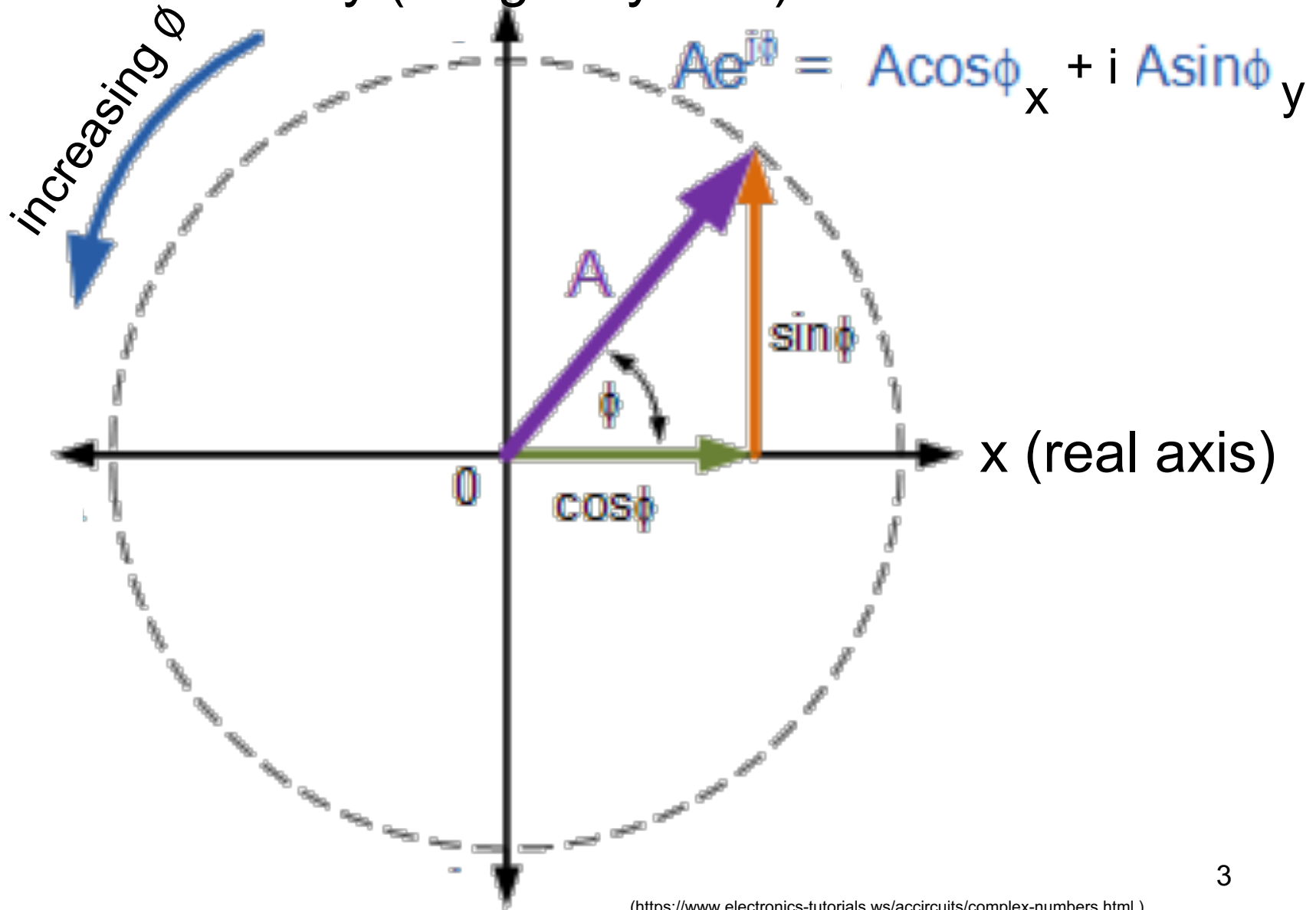
A Complex Number consist of a Real Part and an Imaginary Part



Quantities a and b are real numbers

Use $e^{i\phi} = \cos \phi + i \sin \phi$ (page 682 text)

y (imaginary axis)



Complex numbers

Thus write complex numbers 2 ways:

Consider a complex number: $z = x + i y$

Then z can also be written: $z = |z| e^{i\phi}$

where $|z| = \sqrt{x^2 + y^2}$

$$\tan \phi = y/x$$

Complex number properties:

a) multiplication by a scalar "a" then $a z = a x + i a y$

b) multiplication of complex numbers

$$\begin{aligned} z_1 z_2 &= (x_1 + i y_1) (x_2 + i y_2) \\ &= (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + y_1 x_2) \\ &= |z_1| |z_2| e^{i(\phi_1 + \phi_2)} \end{aligned}$$

Necessary as e.g. H-atom wave functions, Ψ , are complex

First few hydrogen wave functions

- n=1,2 n=3
 Separated
 Combined

$n \quad \ell \quad m_\ell$

$\Psi_{n\ell m_\ell}(r, \theta, \varphi)$

1 0 0 1s

$$\frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

Illustration

2 0 0 2s

$$\frac{1}{4\sqrt{2\pi a_0^3}} \left[2 - \frac{r}{a_0} \right] e^{-r/2a_0}$$

Illustration

2 1 0 2p

$$\frac{1}{4\sqrt{2\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \cos\theta$$

Illustration

2 1 ± 1 2p

$$\frac{1}{8\sqrt{\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0} \sin\theta e^{+i\phi}$$

Illustration

$$a_0 = \frac{\hbar^2}{me^2} = 0.0529 \text{ nm} = \text{first Bohr radius}$$

Complex numbers

Complex number properties (con't):

c) complex conjugate

If $z = x + i y$ then the complex conjugate

$$z^* = x - i y \quad (\text{all } i \rightarrow -i)$$

Then the product $z z^* = (x + i y) (x - i y)$

$$= (x^2 + y^2) + i (-xy + yx)$$

$$= (x^2 + y^2) \text{ a pure real number}$$

This is simpler when we write $z = |z| e^{i\phi}$

because $z^* = |z| e^{-i\phi}$

$$\text{so } z z^* = |z|^2 e^{i0} = |z|^2 = (x^2 + y^2)$$

This will be important when we describe particles as waves: $\Psi(x,t)$. Then the probability to find the particle at position x at time t is $\text{Prob}(x,t) \propto \Psi^* \Psi$

Activity

Evaluate two ways:

$$(4-5i)(12+11i) = z \quad \text{and also} \quad = |z| e^{i\phi}$$

$$(-3-i) - (6-7i)$$

$$(8i)(10+2i)$$

$$(-3-9i)(1+10i)$$

$$(1+5i) / (-3i)$$

$$(7-i) / (2+10i)$$

Activity solution

$$(4-5i)(12+11i) \rightarrow 103 - 16i = 104.2 e^{-i8.8\text{degrees}}$$

$$(-3-i) - (6-7i) \rightarrow -9 + 6i = 10.8 e^{i146.3\text{degrees}}$$

$$(8i)(10+2i) \rightarrow -16 + 80i = 81.6 e^{i101.3\text{degrees}}$$

$$(-3-9i)(1+10i) \rightarrow 87 - 39i = 95.3 e^{-i24.1\text{degrees}}$$

$$(1+5i) / (-3i) \rightarrow -5/3 + (1/3)i = 1.70 e^{i168.7\text{degrees}}$$

$$(7-i) / (2+10i) \rightarrow 1/26 - (9/13)i = 0.69 e^{-i86.8\text{degrees}}$$

If you want more practice go to:

<https://tutorial.math.lamar.edu/problems/alg/complexnumbers.aspx>

Unit vectors and orthonormal unit vectors

It is often convenient to distinguish the direction of a vector from the magnitude of a vector:

write “vector \mathbf{A} ” = $\mathbf{A} = A \hat{\mathbf{A}}$ where A = the magnitude of the vector \mathbf{A} and $\hat{\mathbf{A}}$ is a unit (length) vector in the direction of \mathbf{A} .

It is also convenient to write a vector in terms of its components along e.g. x , y and z directional axes:

$\mathbf{A} = (A_x, A_y, A_z)$ where the component of \mathbf{A} in the x -direction is A_x , etc.

Finally combine the unit vector and component concepts to write $\mathbf{A} = A_x \mathbf{x} + A_y \mathbf{y} + A_z \mathbf{z}$ where \mathbf{x} is a unit vector in the x -direction, etc.

Orthonormal unit vectors \rightarrow basis vectors

To be able to write e.g. a vector as a sum of components replace \mathbf{x} with \mathbf{x}_1 , \mathbf{y} with \mathbf{x}_2 , and \mathbf{z} with \mathbf{x}_3 so that

$\mathbf{A} = A_x \mathbf{x} + A_y \mathbf{y} + A_z \mathbf{z} = \sum A_i \mathbf{x}_i$ with sum from $i = 1$ to 3 ,

$A_i = \mathbf{A} \cdot \mathbf{x}_i$ and the unit vectors are ortho-normal: $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$

When a set of vectors (or functions) span a space, viz all vectors (or functions) can be written in terms of them, then a general e.g. vector can be written:

$\mathbf{A} = \sum A_i \mathbf{x}_i = \sum (\mathbf{A} \cdot \mathbf{x}_i) \mathbf{x}_i$ with in 3 dimensions the summation from $i = 1$ to 3

Activity

Write 2-dimensional vectors, \mathbf{A} , and \mathbf{B} as a sum of components and basis (unit) vectors. Best to write one sum as e.g. $\mathbf{A} = \sum A_i \mathbf{x}_i$ and the other summing over a different index, j . In both sums i and j run from 1 to 2.

Then write the dot product, $\mathbf{A} \cdot \mathbf{B}$, as the dot product between the **two** n -component sums.

Finally what critical relation reduces the two n -component sums to one n -component sum: $\mathbf{A} \cdot \mathbf{B} = \sum A_i B_i$ with i running from 1 to 2.

Activity solution

As suggested: $\mathbf{A} = \sum A_i \mathbf{x}_i$

then $\mathbf{B} = \sum B_j \mathbf{x}_j$ with i and j from 1 to 2 and the coefficients are given by $A_i = \mathbf{A} \cdot \mathbf{x}_i$ and $B_j = \mathbf{B} \cdot \mathbf{x}_j$

So $\mathbf{A} \cdot \mathbf{B} = (\sum A_i \mathbf{x}_i) \cdot (\sum B_j \mathbf{x}_j) = \sum A_i B_j (\mathbf{x}_i \cdot \mathbf{x}_j) = \sum A_i B_j \delta_{ij}$ with the i and j sums over 1 to 2. This operation reminds us that it is OK to move the scalars (A_i and B_j) but need to keep the order of the vectors and the vector “dot” operation.

Finally as $\delta_{ij} = 0$ unless $i=j$ when $\delta_{ii} = \delta_{jj} = 1$ the double sum reduces to a single sum: $\mathbf{A} \cdot \mathbf{B} = \sum A_i B_i$

Note: if you want additional practice do problems at

<https://tutorial.math.lamar.edu/Classes/CalcII/DotProduct.aspx>

<https://tutorial.math.lamar.edu/problems/calci/dotproduct.aspx>

Homework #5

a) Recall that $e^{\pm i\theta} = \cos(\theta) \pm i \sin(\theta)$.

Then using the $\cos(\theta)$ and $\sin(\theta)$ form (for $e^{\pm i\theta}$):

1. What is the complex conjugate of $e^{\pm i\theta} \equiv (e^{\pm i\theta})^*$?
2. Evaluate the product: $e^{\pm i\theta}(e^{\pm i\theta})^*$.

b) Assume that the “ n ” unit vectors: $\hat{x}_i, i = 1, \dots, n$ span a n -dimensional vector space.

In other words any vector \vec{A} can be writing in terms of the \hat{x}_i . This is the simple extension of the unit vectors \hat{x}_1, \hat{x}_2 and \hat{x}_3 being used to write any vector \vec{B} in 3-D:

$\vec{B} = \sum_{j=1}^3 B_j \hat{x}_j$. Using this information:

1. Evaluate the general coefficient B_k in terms of \vec{B} and the unit vectors \hat{x}_k .
2. If we write $\vec{A} = \sum_{i=1}^n A_i \hat{x}_i$, then what is the general coefficient A_k in terms of \vec{A} and the unit vectors \hat{x}_k .
3. write all of the terms for the case $n=2$ to evaluate the dot product: $\mathbf{A} \cdot \mathbf{B} = (\sum A_i \mathbf{x}_i) \cdot (\sum B_j \mathbf{x}_j)$

Orthonormal functions

Can the concept of $\mathbf{A} = \sum A_i \mathbf{x}_i$, where the unit vectors \mathbf{x}_i span the n-dimensional space, be extended to functions so that a general function, $f(x)$ in e.g. the interval $-L \leq x \leq L$, can be written as: $f(x) = \sum A_i \sin(x) + \sum B_j \cos(x)$?

We will find that the quantum mechanical, QM, solutions to a “particle in a box”: $-L/2 \leq x \leq L/2$, are of the form:

$$\begin{aligned}\Psi_n &= A_n \cos(n \pi x / L) \text{ when } n = 1, 3, 5, \dots \\ &= A_n \sin(n \pi x / L) \text{ when } n = 2, 4, 6, \dots\end{aligned}$$

and any general wave function can be written as:

$$\Psi(x) = \sum_{n=1, 3, 5, \dots} A_n \cos(n \pi x / L) + \sum_{n=2, 4, 6, \dots} A_n \sin(n \pi x / L)$$

Orthonormal functions

So how to determine the coefficients, A_n ?

Draw a parallel between writing an arbitrary vector $\mathbf{A} = \sum A_i \mathbf{x}_i$, where the \mathbf{x}_i are an orthonormal set of spanning vectors to $\Psi(x) = \sum_{n=1, 3, 5, \dots} A_n \cos(n \pi x / L) + \sum_{n=2, 4, 6, \dots} A_n \sin(n \pi x / L)$

(checking that) $\cos(n \pi x / L)$ and $\sin(n \pi x / L)$ are an orthonormal set of spanning functions

AND

we need a parallel to the vector dot product operation to determine the coefficients A_n

Practice with Fourier series

Fourier expansion for a function, $f(x)$, on the interval $-L \leq x \leq L$:

$$f(x) = (a_0/2)x1 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)$$

where both sums are for $n = 1$ to ∞

First are $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ orthogonal?

By analogy we want:

$$\cos(m\pi x/L) \text{ “}\cdot\text{” } \cos(n\pi x/L) \propto \delta_{mn}$$

$$\sin(m\pi x/L) \text{ “}\cdot\text{” } \sin(n\pi x/L) \propto \delta_{mn}$$

$$\cos(m\pi x/L) \text{ “}\cdot\text{” } \sin(n\pi x/L) = 0$$

Fourier series

For the operation “ \cdot ” note that:

$$\int_{-L}^L \cos(m\pi x/L) \cos(n\pi x/L) dx = L \delta_{mn}$$

$$\int_{-L}^L \sin(m\pi x/L) \sin(n\pi x/L) dx = L \delta_{mn}$$

$$\int_{-L}^L \underbrace{\cos(m\pi x/L)}_{\text{even function}} \underbrace{\sin(n\pi x/L)}_{\text{odd function}} dx = 0$$

also:

$$1 \times \int_{-L}^L \cos(n\pi x/L) dx = (1) (+L/(n\pi)) \sin(n\pi x/L) \Big|_{-L}^L = 0$$

$$1 \times \int_{-L}^L \sin(n\pi x/L) dx = (1) (-L/(n\pi)) \cos(n\pi x/L) \Big|_{-L}^L = 0$$

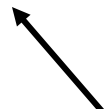
where the latter 2 integrals are zero because $\sin(n\pi) = 0$ in the first case and $\cos(-n\pi) = \cos(n\pi)$ in the second.

Fourier series

So the operation “ \cdot ” resulted in $L \delta_{mn}$

OK orthonormal is nice but is the normal, i.e.
where the operation “ \cdot ” results in δ_{mn} essential?

So we want to write $f(x)$ in “component” functions:
$$f(x) = (a_0/2)x1 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)$$

 the constant term multiplies a function with
no x-dependence which I write as “1”

We now need to do the integral version of the operation “ \cdot ”
on the LHS and on the RHS of this equation for each of
the “component” functions: 1, $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$

First let's recall the vector analog

We want to write an arbitrary vector, \mathbf{A} , in terms of spanning (unit) vectors, \mathbf{x}_i , namely $\mathbf{A} = \sum A_i \mathbf{x}_i$

Because $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$ IF we do a dot product of \mathbf{x}_j on the LHS and the RHS of $\mathbf{A} = \sum A_i \mathbf{x}_i$ what do we get?

LHS: $\mathbf{A} \cdot \mathbf{x}_j$

RHS: $(\sum A_i \mathbf{x}_i) \cdot \mathbf{x}_j = A_j$ because $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$

Then equate LHS = RHS $\rightarrow \mathbf{A} \cdot \mathbf{x}_j = A_j$

Thus the coefficients in the expansion: $\mathbf{A} = \sum A_i \mathbf{x}_i$ are obtained by “dotting” the spanning vector into the vector

\mathbf{A} : $A_i = \mathbf{A} \cdot \mathbf{x}_i$

Now do the same for Fourier series

“component” function = 1

$$\text{LHS} \quad \int_{-L}^L f(x) \cdot 1 \, dx$$

$$\begin{aligned} \text{RHS} \quad & \int_{-L}^L [(a_0/2) \cdot 1 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)] \cdot 1 \, dx \\ & = (a_0/2) \times (2L) = L a_0 \end{aligned}$$

Equating RHS = LHS we get

$$a_0 = (1/L) \times \int_{-L}^L f(x) \, dx$$

$$\text{thus the term in the expansion } a_0/2 = \frac{\int_{-L}^L f(x) \, dx}{2L} = \langle f(x) \rangle$$

So the constant term in the expansion is just the average of the function, $f(x)$, in the interval $-L \leq x \leq L$

Fourier series

“component” function = $\cos(m\pi x/L)$

$$\text{LHS} \quad \int_{-L}^L f(x) \cos(m\pi x/L) dx$$

$$\begin{aligned} \text{RHS} \quad & \int_{-L}^L [(a_0/2) 1 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)] \cos(m\pi x/L) dx \\ & = \sum a_n (L \delta_{nm}) \\ & = L a_m \end{aligned}$$

Equating the RHS = LHS we get:

$$a_m = (1/L) \times \int_{-L}^L f(x) \cos(m\pi x/L) dx$$

Fourier series

“component” function = $\sin(m\pi x/L)$

$$\text{LHS} \quad \int_{-L}^L f(x) \sin(m\pi x/L) dx$$

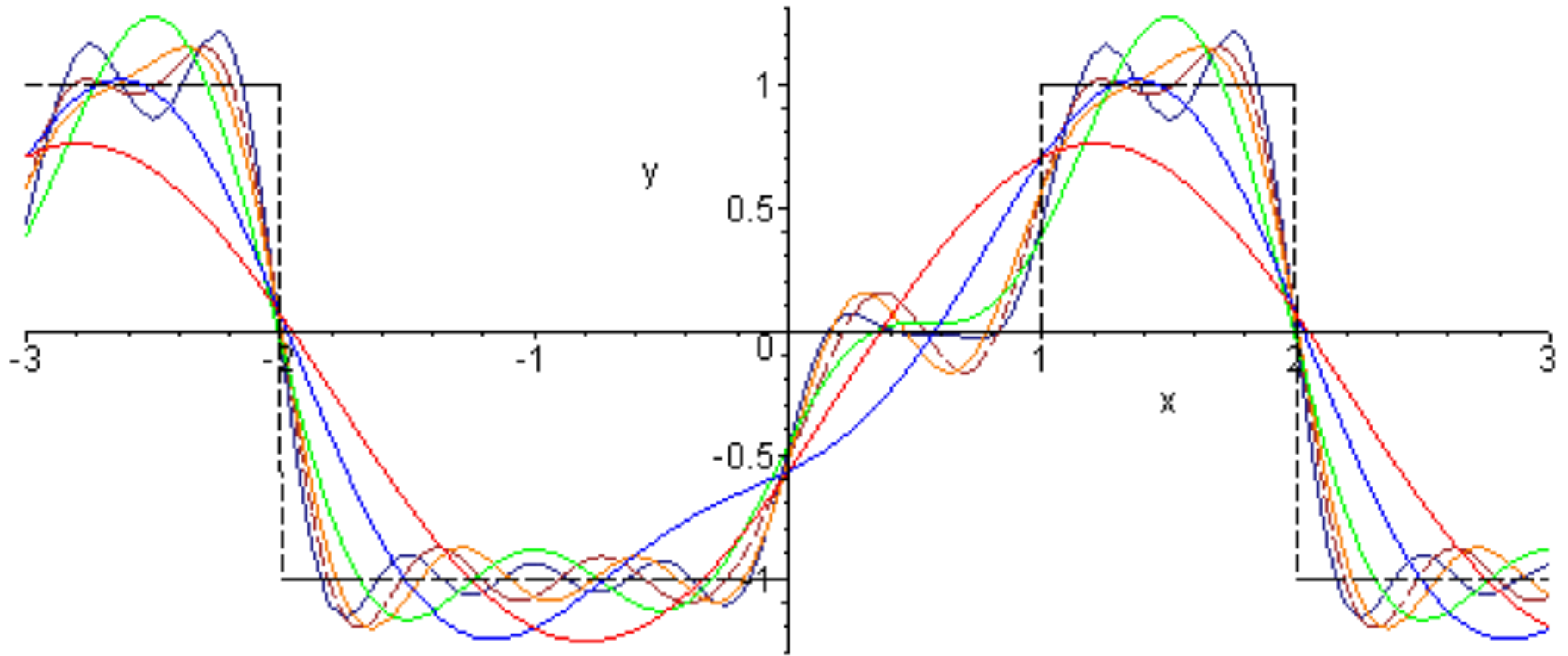
$$\begin{aligned} \text{RHS} \quad & \int_{-L}^L [(a_0/2) 1 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)] \sin(m\pi x/L) dx \\ & = \sum b_n (L \delta_{nm}) \\ & = L b_m \end{aligned}$$

Equating the RHS = LHS we get:

$$b_m = (1/L) \times \int_{-L}^L f(x) \sin(m\pi x/L) dx$$

Many examples on the Internet

Each successive curve shows the sum of all terms up to that term



To see some worked out problems go to:

http://exampleproblems.com/wiki/index.php/Fourier_Series

then click “[solution](#)”

Activity

Expand $f(x) = 3.5 \cos(16 \pi x / L)$ in a Fourier series:

$$f(x) = (a_0/2)x1 + \Sigma a_n \cos(n\pi x/L) + \Sigma b_n \sin(n\pi x/L)$$

Use the integral version of the operation “ \cdot ” on the LHS and on the RHS of this equation for each of the Fourier “component” functions: **1**, **$\cos(m\pi x/L)$** and **$\sin(m\pi x/L)$**

Activity useful relations

Expand $f(x) = 3.5 \cos(16 \pi x / L)$ in a Fourier series:

$$f(x) = (a_0/2)x1 + \Sigma a_n \cos(n\pi x/L) + \Sigma b_n \sin(n\pi x/L)$$

Remember that:

$$\int_{-L}^L \cos(m\pi x/L) \cos(n\pi x/L) dx = L \delta_{mn}$$

$$\int_{-L}^L \sin(m\pi x/L) \sin(n\pi x/L) dx = L \delta_{mn}$$

$$\int_{-L}^L \cos(m\pi x/L) \sin(n\pi x/L) dx = 0$$

Also that:

$$a_0 = (1/L) \times \int_{-L}^L f(x) dx$$

$$a_m = (1/L) \times \int_{-L}^L f(x) \cos(m\pi x/L) dx$$

$$b_m = (1/L) \times \int_{-L}^L f(x) \sin(m\pi x/L) dx$$

Activity solution

$$\begin{aligned} a_0 / 2 &= (1 / (2 L)) \times \int_{-L}^L 3.5 \cos(16 \pi x / L) dx \\ &= (1 / (2 L)) \times 3.5 \times (L / (16 \pi)) \sin(16 \pi x / L) \Big|_{-L}^L = 0 \end{aligned}$$

$$\begin{aligned} a_m &= (1/L) \times \int_{-L}^L 3.5 \cos(16\pi x/L) \cos(m\pi x/L) dx \\ &= (1/L) \times (3.5 L \delta_{16 m}) \rightarrow a_{16} = 3.5 \text{ all others are } 0 \end{aligned}$$

$$\begin{aligned} b_m &= (1/L) \times \int_{-L}^L 3.5 \cos(16\pi x/L) \sin(m\pi x/L) dx \\ &= 0 \end{aligned}$$

So the expansion:

$$f(x) = a_0/2 + \sum a_n \cos(n\pi x/L) + \sum b_n \sin(n\pi x/L)$$

$$\begin{aligned} \text{has only one term, } f(x) &= a_{16} \cos(16\pi x/L) \\ &= 3.5 \cos(16\pi x/L) \end{aligned}$$

Homework #6

Assume that the function, $f(x)$ can be written in a Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

for $f(x)$ defined on the interval $-L \leq x \leq L$.

a) If $f(x)$ is an *odd* function of x : $f(-x) = -f(x)$, show that $a_n = 0$ for all $n \geq 0$.

Hint: $\int_{-L}^0 g(x)dx = \int_0^L g(-x)dx$.

b) If $f(x)$ is an *even* function of x : $f(-x) = f(x)$, show that $b_n = 0$ for all $n \geq 1$.

c) If the average value of $f(x) \equiv \langle f(x) \rangle = \frac{\int_{-L}^L f(x)dx}{\int_{-L}^L dx}$ then show that:

$$a_0 = 2 \langle f(x) \rangle.$$

Orthonormal unit vectors \rightarrow basis vectors

To be able to write e.g. a vector as a sum of components replace \mathbf{x} with \mathbf{x}_1 , \mathbf{y} with \mathbf{x}_2 , and \mathbf{z} with \mathbf{x}_3 so that $\mathbf{A} = A_x \mathbf{x} + A_y \mathbf{y} + A_z \mathbf{z} = \sum A_i \mathbf{x}_i$ with sum from $i = 1$ to 3 and $A_i = \mathbf{A} \cdot \mathbf{x}_i$.

It is convenient to generalize this to “n” dimensions with “n” unit vectors, \mathbf{x}_i , along the directions of the n-orthogonal axes: thus $\mathbf{x}_i \cdot \mathbf{x}_j = \delta_{ij}$.

When a set of vectors (or functions) span a space, viz all vectors (or functions) can be written in terms of them, then a general e.g. vector can be written:

$$\mathbf{A} = \sum A_i \mathbf{x}_i = \sum (\mathbf{A} \cdot \mathbf{x}_i) \mathbf{x}_i \quad \text{with the summation from } i = 1 \text{ to } n$$